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Relations in the homotopy of simplicial abelian Hopf algebras

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Abstract

In this paper, we analyze the structure possessed by the homotopy groups of a simplicial abelian Hopf algebra over the field $\mathbb{F}_2$. Specifically, we review the higher-order structure that the homotopy groups of a simplicial commutative algebra and simplicial cocommutative coalgebra possess. We then demonstrate how these structures interact under the added conditions present in a Hopf algebra. © 1999 Elsevier Science B.V. All rights reserved.

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0. Introduction

The goal of this paper is to determine all the natural relations among primary operations that occur in the homotopy of a simplicial abelian Hopf algebra over $\mathbb{F}_2$, the field of two elements. Here abelian Hopf algebra means a unitary commutative algebra in the category of counitary cocommutative coalgebras.

The motivation for this problem comes studying the second quadrant cohomology spectral scquence, over a finite field, associated to a cosimplicial space (see [3]). The E_2-term of such a spectral sequence is the homotopy groups for the simplicial (unstable) algebra obtained from applying cohomology to this cosimplicial space. Given a simplicial commutative $\mathbb{F}_2$-algebra $A_.$ its homotopy groups possess operations

$$\delta_i : \pi_n A_. \to \pi_{n+i} A_.$$

as constructed in [4, 2, 9]. As such they are called *Cartan–Bousfield–Dwyer operations.* In [10] it is shown that if $A_.$ arises from cohomology, as above, then, in the associated

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spectral sequence, these operations guarantee that Steenrod operations, which would violate instability at E_∞, do not survive.

Now if one started off with a cosimplicial iterated loop space, the associated cohomology is also a simplicial cocommutative coalgebra. The homotopy of such an object $B_.$ has a right action of Steenrod operations

$$Sq^i : \pi_n B_. \to \pi_{n-i} B_.$$

which can be extracted from [7]. If $B_.$ arises from cohomology, as above, then these Steenrod operations determine Dyer–Lashof operations in the abuttment. This is shown in [19, 20] examines a specific example.

Now the cohomology of a cosimplicial iterated loop space is more than just a simplicial algebra and a simplicial coalgebra, it is a simplicial Hopf algebra. Given such an object $H_.$ the added Hopf condition guarantees certain "Nishida relations" amongst the above operations, in the homotopy. The determination of these relations is the objective of this paper.

We organize this paper as follows. In Section 1, we review the primary structure possessed by the homotopy of simplicial commutative algerbras and simplicial cocommutative coalgebras. This is codified through the notions of *D-algebra* and *A-coalgebra* respectively. We then define *Hopf D-algebra* and state the main theorem that says this correctly describes the relations that occur in the homotopy of a simplicial abelian Hopf algebra. In Section 2, we describe a natural map between composite functors of vector spaces. This natural map is useful in describing a modified Hopf condition for abelian Hopf algebras. We also describe a simple relation that occurs in a Hopf Γ-algebra. In Section 3, we review the homotopy of the symmetric (co)-invariants on a simplicial vector space. We use this, in Section 4, to reduce the proof of the main theorem to showing certain relations occur in the homotopy of the symmetric invariants on any simplicial commutative algebra. This is reduced further to computing the effect, in homotopy, of the natural map of Section 2, prolonged to simplicial vector spaces. In Section 5, we show that such a calculation is determined by the effect, in homotopy, of a natural idempotent on the D_8-invariants acting on the fourfold tensor product of a simplicial vector space (here D_8 is the dihedral group of order 8, viewed as a subgroup of the symmetric group on four letters). In Section 6, we review some needed tools from group cohomology plus a useful theorem from [9] which allows us to prove our final reduction to a group cohomological calculation. This calculation is performed in Section 7.

1. Simplicial algebras, coalgebras, Hopf algebras, and their homotopy

In this section, we review the structure of the homotopy groups for simplicial commutative algebras and simplicial cocommutative coalgebras. This is expressed through the notions of D-algebras and A-coalgebras, respectively. As an immediate consequence the homotopy groups of a simplicial bicommutative Hopf algebra possesses both of these structures which the Hopf condition tells us must interact in a meaningful way. We codify through the notion of a Hopf D-algebra. The main theorem of this paper is that this notion accurately describes this expanded structure.

We denote the category of commutative algebras (respectively commutative graded algebras) by $\mathscr{A}$ (respectively $\mathscr{A}_*$).

Given a graded algebra Λ, let $I_s(\Lambda) \subseteq \Lambda s \geq 0$ denote the ideal of elements x in Λ such that $|x| \geq s$.

Definition 1.1. A *Γ-algebra* is a commutative graded algebra Λ together with a map

$$\gamma_2 : \Phi I_2 \to \Lambda$$

such that

1. I_1 is exterior under the product of Λ,
2. for $x, y \in I_2$

$$\gamma_2(\overline{x \cdot y}) = \gamma_2(\bar{x}) + \gamma_2(\bar{y}) + x \cdot y,$$

3. for $x, y \in \Lambda$ such that $x \cdot y \in I_2$

$$\gamma_2(\overline{x \cdot y}) = \begin{cases} 0, & x, y \in I_1, \\ (x \cdot x) \cdot \gamma_2(\bar{y}), & |x| = 0, \\ \gamma_2(\bar{x}) \cdot (y \cdot y), & |y| = 0. \end{cases}$$

We now make the following, as given in [12].

Definition 1.2. A *D-algebra* Λ is a Γ-algebra together with maps

$$\delta_i : \Lambda_n \to \Lambda_{n+i}$$

for all $2 \leq i \leq n$ such that

1. δ_i is a homomorphism, for $i < n$, and $\delta_n = \gamma_2$,
2. for $x, y \in \Lambda$ such that $x \cdot y \in \Lambda_n$ then

$$\delta_i(x \cdot y) = \begin{cases} (x \cdot x) \cdot \delta_i(y), & |x| = 0, \\ \delta_i(x) \cdot (y \cdot y), & |y| = 0, \\ 0 & \text{otherwise}, \end{cases}$$

3. for $x \in \Lambda_n$ and $j < 2i$ then

$$\delta_j \delta_i x = \sum_{\frac{j+1}{2} \leq s \leq \frac{i+j}{3}} \binom{i - j + s - 1}{i - s} \delta_{j+i-s} \delta_s x.$$

A map of D-algebras is a map in $\mathscr{A}_*$ that commutes with the δ_i. We denote the category of D-algebras by $\mathscr{A}\mathscr{D}$.

We denote the category of simplicial commutative algebras by s$\mathscr{A}$.

The following was proved in [9, 13].

Theorem 1.3. *Let A be a simplicial commutative algebra. Then $\pi_* A$ is naturally a D-algebra, i.e. we have a functor*

$$\pi_* : \mathrm{s}\mathscr{A} \to \mathscr{A}\mathscr{D}.$$

Remark. The operations δ_i in (1.2) were first discovered in [4]. Their properties were subsequently derived in [2, 9]. In the latter, they were called *higher divided squares*. We will also refer to them as *Cartan–Bousfield–Dwyer operations*.

Next, denote the category of cocommutative coalgebras (resp. cocommutative graded coalgebras) by $\mathscr{C}\mathscr{A}$ (resp. $\mathscr{C}\mathscr{A}_*$).

Given a cocommutative graded coalgebra Π we define the coalgebra map

$$\xi : \Pi \to \Phi\Pi \tag{1.4}$$

called the Verschiebung, as the linear dual of the squaring map (we will give a more meaningful definition in the next section).

Definition 1.5. An *A-coalgebra* is a cocommutative graded coalgebra Π together with homomorphisms

$$Sq^i : \Pi_n \to \Pi_{n-i}$$

for $i \geq 0$ such that for $x \in \Pi_n$ we have

1. $xSq^i = 0$ for $2i > n$ and $xSq^{n/2} = \xi(x)$,
2. if $\Delta x = \Sigma x' \otimes x''$ then
$$\Delta(xSq^i) = \sum_{s+t=i} \sum (x'Sq^s) \otimes (x''Sq^t),$$
3. for $j < 2i$ we have
$$xSq^j Sq^i = \sum_{2s \leq j} \binom{i-s-1}{j-2s} xSq^{i+j-s} Sq^s.$$

We define a map of A-coalgebras to be a map in $\mathscr{C}\mathscr{A}_*$ which commutes with the Sq^i. Denote by $\mathscr{K}^*$ the category of A-coalgebras.

Note. A clearly denotes the Steenrod algebra.

We denote the category of simplicial cocommutative coalgebras by $\mathrm{s}\mathscr{C}\mathscr{A}$. A consequence of [7] (see also [12]) is the following

Theorem 1.6. *Let Π be a simplicial cocommutative coalgebra. Then $\pi_* \Pi$ is naturally an A-coalgebra. That is, we have a functor*

$$\pi_* : \mathrm{s}\mathscr{C}\mathscr{A} \to \mathscr{K}^*.$$

Recall, now, that a (graded) Hopf algebra (in the sense of [17]) is a (graded) module H which is both a (graded) algebra and a (graded) coalgebra for which the two diagrams

$$\begin{tikzcd} H\otimes H \arrow[r, "\Delta\otimes\Delta"] \arrow[dd, "m"'] & H\otimes H\otimes H\otimes H \arrow[d, "1\otimes T\otimes 1"] \\ & H\otimes H\otimes H\otimes H \arrow[d, "m\otimes m"] \\ H \arrow[r, "\Delta"'] & H\otimes H \end{tikzcd} \tag{1.7}$$

and

$$\begin{tikzcd} \mathbb{F}_2 \arrow[r, "\eta"] \arrow[dr, "1"'] & H \arrow[d, "\varepsilon"] \\ & \mathbb{F}_2 \end{tikzcd} \tag{1.8}$$

commute. A map of Hopf algebras is simply a map of algebras and a map of coalgebras. We further define a Hopf algebra to be abelian if it is commutative as an algebra and cocommutative as a coalgebra.

We denote by $\mathscr{HA}$ (resp. $\mathscr{HA}_*$) the category of abelian Hopf algebras (resp. abelian graded Hopf algebras).

Definition 1.9. A *Hopf Γ-algebra* is a pair (H,γ_2) consisting of an abelian graded Hopf algebra H together with a map

$$\gamma_2 : \Phi I_2 \to H$$

satisfying 1–3 of Definition 1.1 along with the additional condition 4 for $x\in I_2$

$$\Delta\gamma_2\bar{x} = \gamma_2(\overline{\Delta x}).$$

A map of Hopf Γ-algebras is just a map in $\mathscr{H}_*$ which is also a map of Γ-algebras.

Definition 1.10. A *Hopf D-algebra* is a Hopf Γ-algebra H together with maps

$$\delta_i : H_n \to H_{n+i}$$

for all $2\le i\le n$, satisfying conditions 1–3 of Definition 1.2, and with maps

$$Sq^i : H_n \to H_{n-i}$$

for all $i\ge 0$, satisfying conditions 1–3 of Definition 1.5, such that the following relations are satisfied for a fixed $x\in H_n$:

1. for each $2 \le i \le n$

$$\Delta\delta_i x = \delta_i \Delta x$$

and for any $y \in H$, $j \ge 0$

$$(x \cdot y)Sq^j = \sum_{s+t=j} (xSq^s) \cdot (ySq^t),$$

2. for each $2 \le j < n$ and $i \ge 0$

$$(\delta_j x)Sq^i = \begin{cases} \sum_s (i-j,\ j-2i+2s-1)\delta_{j-i+s}(xSq^s), & i > j, \\ \xi\gamma_2 x + \sum_{2s>j} \delta_s(xSq^s) & i = j, \\ \sum_s (i-2s,\ j-2i+2s-1)\,\delta_{j-i+s}(xSq^s), & i < j, \end{cases}$$

$$(\delta_n x)Sq^i = \begin{cases} 0, & i > n, \\ \xi\gamma_2 x, & i = n, \\ \sum_{2s<i} (xSq^s)(xSq^{i-s}) & \\ \quad + \sum_s (i-2s,\ n-2i+2s-1)\delta_{n-i+s}(xSq^s), & i < n. \end{cases}$$

A map of Hopf D-algebras is simply a map of D-algebras and a map of A-coalgebras. We denote the category of Hopf D-algebras by $\mathscr{HD}$.

We denote by $s\mathscr{HA}$ the category of simplicial abelian Hopf algebras.

We now come to the main theorem of this work, whose proof is postponed to Section 4.

Theorem 1.11. *Let H be a simplicial abelian Hopf algebra. Then $\pi_* H$ is naturally a Hopf D-algebra. That is we have a functor*

$$\pi_* : s\mathscr{HA} \to \mathscr{HD}.$$

We close this section by noting some consequences of this theorem and of the notion a Hopf D-algebra in general. These will be proved in [14].

Since objects of $\mathscr{AD}$ are augmented commutative $\mathbb{F}_2$-algebras, the indecomposables functor defines a functor

$$Q : \mathscr{AD} \to \mathscr{UD}$$

where the target is the category of *unstable D-modules*. Let $\mathscr{B}$ be the algebra of Cartan–Bousfield–Dwyer operations. Then $\mathscr{UD}$ is the category of unstable $\mathscr{B}$-modules and we have a functor

$$\mathbb{F}_2 \otimes_{\mathscr{B}} (-) : \mathscr{UD} \to m\mathbb{F}_2.$$

Let

$$Q_D : \mathscr{A}\mathscr{D} \to m\mathbb{F}_2$$

be the composite of these two functors. By restriction we have a functor

$$Q_D : \mathscr{H}\mathscr{D} \to \mathscr{L}$$

where $\mathscr{L}$ is the category of right $\mathscr{A}_0$-modules, $\mathscr{A}_0 = \mathbb{F}_2[Sq^0, Sq^1]/[(Sq^1)^2]$. This follows from 2 of Definition 1.10. Let $\mathscr{H}\mathscr{D}_+$ be the subcategory of *connected* objects in $\mathscr{H}\mathscr{D}$ (i.e. isomorphic to $\mathbb{F}_2$ in degree 0) and $\mathscr{L}_+$ be the subcategory of *connected* objects in $\mathscr{L}$ (i.e. 0 in degree 0).

Theorem 1.12. *The functor $Q_D : \mathscr{H}\mathscr{D} \to \mathscr{L}$ possesses a right adjoint*

$$\Lambda : \mathscr{L} \to \mathscr{H}\mathscr{D}$$

such that the restricted adjoint pair

$$Q_D : \mathscr{H}\mathscr{D}_+ \Leftrightarrow \mathscr{L}_+ : \Lambda$$

is an equivalence of categories.

An interesting input to the proof of this theorem is

Proposition 1.13. *If $H \in \mathscr{H}\mathscr{D}_+$ then H is a free D-algebra.*

To make this useful for simplicial abelian Hopf algebras we prove

Theorem 1.14. *For H π_0-connected in* s$\mathscr{H}\mathscr{A}$ *we have*

$$Q_D(\pi_* H) \cong H_*^Q(H)$$

where the right-hand side is the André-Quillen homology of H as a simplicial algebra (see [12]).

As an application, let A be a connected simplicial abelian group. Then $\mathbb{F}_2[A]$ is a connected object of s$\mathscr{H}\mathscr{A}$ so from these results

$$H_*(A; \mathbb{F}_2) \cong \pi_* \mathbb{F}_2[A] \cong \Lambda(H_*^Q(\mathbb{F}_2[A]))$$

and we have a short exact sequence

$$0 \to \mathbb{Z}/2 \otimes \pi_q A \to H_q^Q(\mathbb{F}_2[A]) \to Tor(\mathbb{Z}/2, \pi_{q-1} A) \to 0.$$

This recovers H. Cartan's main computation in [4].

2. Group actions and consequences of commutativity

Let V be an $\mathbb{F}_2$-module and define

$$V^{\otimes m} = \underbrace{V \otimes \cdots \otimes V}_{m\text{-times}}.$$

Then Σ_m, the symmetric group on m letters, acts on $V^{\otimes m}$ by permutation. Thus, for any subgroup $G \leq \Sigma_m$, $V^{\otimes m}$ is a G-module. With this we define the G-symmetric invariant functor

$$S^G : m\mathbb{F}_2 \to m\mathbb{F}_2 \tag{2.1}$$

by

$$S^G(V) = (V^{\otimes m})^G$$

and the G-symmetric coinvariant functor

$$S_G : m\mathbb{F}_2 \to m\mathbb{F}_2 \tag{2.2}$$

by

$$S_G(V) = (V^{\otimes m})_G.$$

If $G = \Sigma_m$ then we denote (2.1) by S^m and (2.2) by S_m.

Now, let $\overline{N} \in \mathbb{F}_2[G]$ be defined by

$$\overline{N} = \sum_{g \in G} g. \tag{2.3}$$

Then the action of $\overline{N}$ on $V^{\otimes m}$ defines a map which factors

$$\begin{tikzcd} V^{\otimes m} \arrow[rr, "\overline{N}"] \arrow[dr, "\tau"'] & & V^{\otimes m} \\ & S^G V \arrow[ur, "i"'] & \end{tikzcd} \tag{2.4}$$

but since, for any $x \in V^{\otimes m}$, $\tau(gx) = \tau(x)$ for any $g \in G$ then we have a further factorization,

$$\begin{tikzcd} V^{\otimes m} \arrow[rr, "\tau"] \arrow[dr, "\rho"'] & & S^G V \\ & S_G V \arrow[ur, "N"'] & \end{tikzcd} \tag{2.5}$$

defining the norm map N.

Because of its importance later, we analyze the norm map N in the case $G = \Sigma_2$. First, we define the diagonal map

$$d : \Phi V \to V^{\otimes 2}$$

by $d\bar{x} = x \otimes x$. This is not a homomorphism, nonetheless we have a commuting diagram

$$\begin{tikzcd} & S^2V \arrow[d, "i"] \\ \Phi V \arrow[ur, "\sigma"] \arrow[r, "d"] \arrow[dr, "\iota"'] & V^{\otimes 2} \arrow[d, "\rho"] \\ & S_2V \end{tikzcd} \tag{2.6}$$

σ is not a homomorphism, but ι is one. From this we define the exterior square functor E_2 by $E_2V = \operatorname{coker} \iota$. We now have the following commutative diagram:

$$\begin{tikzcd} & & & & \Phi V \arrow[d, "\sigma"'] \arrow[dr, "\cong"] & & \\ 0 \arrow[r] & \Phi V \arrow[r, "\iota"] & S_2V \arrow[rr, "N"] \arrow[dr, "\zeta"'] & & S^2V \arrow[r, "\pi"] & \operatorname{coker} N \arrow[r] & 0 \\ & & & E_2V \arrow[ur, "v"'] & & & \end{tikzcd} \tag{2.7}$$

from which we have that $E_2V = \operatorname{im} N = \ker \pi$. Note that $\pi\sigma$ is a linear isomorphism.

As an application of (2.7) we have

Proposition 2.8. *For any $\omega \in S^2V$ there exists $\alpha \in E_2V$ and $x \in V$, uniquely determined by ω, such that*

$$\omega = v(\alpha) + \sigma(\bar{x}).$$

Proof. Let $\tilde{\sigma} : \operatorname{coker} N \to S^2V$ be the composite $\sigma \cdot (\pi\sigma)^{-1}$ so that $\pi\tilde{\sigma} = 1$. Then the self-map

$$1 + \tilde{\sigma}\pi : S^2V \to S^2V$$

satisfies $\pi(1 + \tilde{\sigma}\pi) = 0$. Thus since v is injective, there exists $\alpha \in E_2V$ such that

$$v(\alpha) = \omega + \tilde{\sigma}\pi(\omega).$$

Finally, let $x \in V$ be the element which satisfies

$$\bar{x} = (\pi\sigma)^{-1}(\pi\omega).$$

The conclusion follows. $\square$

Now, given a commutative algebra Λ the product map

$$m : \Lambda \otimes \Lambda \to \Lambda, \tag{2.9}$$

factors as

$$\begin{tikzcd} \Lambda\otimes\Lambda \arrow[rr, "m"] \arrow[dr, "\rho"'] & & \Lambda \\ & S_2\Lambda \arrow[ur, "\mu"'] & \end{tikzcd} \tag{2.10}$$

Similarly, given a cocommutative coalgebra Π the coproduct map

$$\Delta:\Pi\to\Pi\otimes\Pi, \tag{2.11}$$

factors as

$$\begin{tikzcd} \Pi \arrow[rr, "\Delta"] \arrow[dr, "\psi"'] & & \Pi\otimes\Pi \\ & S^2\Pi \arrow[ur, "i"'] & \end{tikzcd} \tag{2.12}$$

From this we now proceed to prove

Proposition 2.13. *For an abelian Hopf algebra H, there is a natural map of modules*

$$\phi : S_2S^2V \to S^2S_2V$$

such that the following diagram commutes:

$$\begin{tikzcd} S_2H \arrow[r, "S_2(\psi)"] \arrow[d, "\mu"'] & S_2S^2H \arrow[r, "\phi"] & S^2S_2H \arrow[d, "S^2(\mu)"] \\ H \arrow[rr, "\psi"] & & S^2H \end{tikzcd}$$

To prove this we need the following lemmas.

Lemma 2.14. *Let V be a (graded) module. Then there exist maps ϕ', ϕ'' of modules such that the following diagram commutes:*

$$\begin{tikzcd} S^2V\otimes S^2V \arrow[r, "i\otimes i"] \arrow[d, "\phi'"'] & V\otimes V\otimes V\otimes V \arrow[r, "\rho"] \arrow[d, "1\otimes T\otimes 1"] & S_2(V\otimes V) \arrow[d, "\phi''"] \\ S^2(V\otimes V) \arrow[r, "i"'] & V\otimes V\otimes V\otimes V \arrow[r, "\rho\otimes\rho"'] & S_2V\otimes S_2V \end{tikzcd}$$

Proof. Define $\widehat{T}:V^{\otimes 4}\to V^{\otimes 4}$ by $\widehat{T}(a\otimes b\otimes c\otimes d)=c\otimes d\otimes a\otimes b$. Then on $V^{\otimes 4}$, we have the identity

$$(1\otimes T\otimes 1)(T\otimes T)=\widehat{T}(1\otimes T\otimes 1).$$

From this, the two composites

$$S^2V \otimes S^2V \xrightarrow{i \otimes i} V^{\otimes 4} \xrightarrow{1 \otimes T \otimes 1} V^{\otimes 4}$$

and

$$V^{\otimes 4} \xrightarrow{1 \otimes T \otimes 1} V^{\otimes 4} \xrightarrow{\rho \otimes \rho} S_2V \otimes S_2V$$

factors to give us the maps ϕ' and ϕ'', respectively. □

Lemma 2.15. *For a module V there exists a map ϕ of modules such that the following cube commutes*

$$\begin{tikzcd}
S^2V \otimes S^2V \arrow[rr, "i\otimes i"] \arrow[dr, "\rho"] \arrow[dd, "\phi'"'] & & V^{\otimes 4} \arrow[dr, "\rho"] \arrow[dd, "1\otimes T\otimes 1"' near start] & \\
& S_2S^2V \arrow[rr, "S_2(i)" near start] \arrow[dd, "\phi" near start] & & S_2(V\otimes V) \arrow[dd, "\phi''"] \\
S^2(V\otimes V) \arrow[rr, "i" near start] \arrow[dr, "S^2(\rho)"'] & & V^{\otimes 4} \arrow[dr, "\rho\otimes\rho"] & \\
& S^2S_2V \arrow[rr, "i"'] & & S_2V \otimes S_2V
\end{tikzcd}$$

Proof. The identity $(1 \otimes T \otimes 1)(T \otimes T) = \widehat{T}(1 \otimes T \otimes 1)$ from the proof of Lemma (1.7), tells us that the composite

$$S^2V \otimes S^2V \xrightarrow{\phi'} S^2(V \otimes V) \xrightarrow{S^2(\rho)} S^2S_2V$$

factors to give us the desired map ϕ. The commutativity of the cube now follows from Lemma 2.14, the surjectivity of ρ, and the injectivity of i. □

Proof of Proposition 2.13. The diagram (1.7) can be expanded to give

$$\begin{tikzcd}
H\otimes H \arrow[rrrr, "\Delta\otimes\Delta"] \arrow[drr, "\psi\otimes\psi"] \arrow[ddr, "\rho"'] \arrow[ddddd, "m"'] & & & & H\otimes H\otimes H\otimes H \arrow[d, "1\otimes T\otimes 1"] \\
& & S^2H\otimes S^2H \arrow[urr, "i\otimes i"'] \arrow[d, "\rho"] & & H\otimes H\otimes H\otimes H \arrow[ddl, "\rho\otimes\rho"'] \arrow[dddd, "m\otimes m"] \\
& S_2H \arrow[r, "S_2(\psi)"] \arrow[dddl, "\mu"'] & S_2S^2H \arrow[d, "\phi"] & & \\
& & S^2S_2H \arrow[r, "i"] \arrow[d, "S^2(\mu)"] & S_2H\otimes S_2H \arrow[ddr, "\mu\otimes\mu"] & \\
& & S^2H \arrow[drr, "i"] & & \\
H \arrow[urr, "\psi"] \arrow[rrrr, "\Delta"'] & & & & H\otimes H
\end{tikzcd}$$

which commutes by (2.10), (2.12), Lemma 2.15, the surjectivity of ρ, and the injectivity of i. $\square$

We now pause to give a useful reinterpretation of Proposition 2.13.

Let Λ be a commutative algebra. Then $S^2\Lambda$ is a commutative algebra with product

$$S_2S^2\Lambda \xrightarrow{\phi} S^2S_2\Lambda \xrightarrow{S^2(\mu)} S^2\Lambda \tag{2.16}$$

and unit

$$\mathbb{F}_2 \simeq S^2(\mathbb{F}_2) \xrightarrow{S^2(\eta)} S^2\Lambda. \tag{2.17}$$

Corollary 2.18. *For an abelian Hopf algebra H, the coproduct*

$$\psi : H \to S^2H$$

is a map of commutative algebras.

Also, if Λ is a Γ-algebra then $\Lambda \otimes \Lambda$ is a Γ-algebra. Moreover, from its definition we have

$$\gamma_2 T = T\gamma_2. \tag{2.19}$$

Thus $S^2\Lambda$ is also a Γ-algebra.

Now, as an application of Proposition 2.8 we give another interpretation of the Verschiebung for a cocommutative coalgebra Π. Fix $x \in \Pi$. Then $\psi x \in S^2\Pi$. By Proposition 2.8 there exists unique $\alpha \in E_2\Pi$ and $\beta \in \Pi$ such that

$$\psi x = \nu(\alpha) + \sigma(\bar{\beta}).$$

From this we can set $\xi(x) = \bar{\beta}$.

We can use this to record a basic relation on a Hopf Γ-algebra H. Our objective is to give a description of the composite

$$\Phi I_2 \xrightarrow{\gamma_2} H \xrightarrow{\xi} \Phi H.$$

To do so we define a map

$$h : I_2 \to H \tag{2.20}$$

which fits in the following expansion of (2.7)

$$\begin{tikzcd}
 & & & & \Phi H \arrow[d, "\sigma"'] \arrow[dr, "\cong"] & & \\
0 \arrow[r] & \Phi H \arrow[r, "i"] & S_2H \arrow[rr, "N"] \arrow[dr, "\zeta"] \arrow[dd, "\mu"'] & & S^2H \arrow[r, "\pi"] \arrow[d, "\alpha"] & \operatorname{coker} N \arrow[r] & 0 \\
 & & & E_2H \arrow[ur, "v"] \arrow[r, equal] \arrow[dl, dashed] & E_2H & & \\
 & & H & I_2 \arrow[l, "h"] \arrow[rr, hook] & & H \arrow[uul, "\psi"'] &
\end{tikzcd}$$

Here α is the natural map determined by Proposition 2.8 and the dotted arrow exists in positive degrees by 1 of Definition 1.9.

We can give an explicit description of the map h as follows: for $x \in H$ write

$$\psi x = \Sigma(x' \otimes x'' + x'' \otimes x') + \Sigma y \otimes y$$

then

$$h(x) = \Sigma x'x''. \tag{2.21}$$

Proposition 2.22. *For a Hopf Γ-algebra H the diagram*

$$\begin{tikzcd}
\Phi I_2 \arrow[r, "\gamma_2"] \arrow[dr, "\Phi h"'] & H \arrow[d, "\xi"] \\
 & \Phi H
\end{tikzcd}$$

commutes.

To prove this, we note that since $i: S^2H \to H \otimes H$ is a map of Γ-algebras then $\psi: H \to S^2H$ is a map of Γ-algebras, by Corollary 2.18 and Definition 1.9. In light of this and Proposition 2.8 we are reduced to proving

Lemma 2.23. *Let A be a Γ-algebra and $\omega \in S^2A$. Write $\omega = v(\alpha) + \sigma(\bar{\beta})$ as in Proposition 2.8. Then*

$$\pi(\gamma_2\omega) = \pi\sigma(\mu(\overline{\alpha}))$$

where $\overline{\alpha} \in S_2A$ satisfies $\zeta(\overline{\alpha}) = \alpha$.

Proof. Since γ_2 is quadratic, we have

$$\gamma_2\omega = \gamma_2 v(\alpha) + \gamma_2\sigma(\bar{\beta}) + v(\alpha) \cdot \sigma(\bar{\beta}).$$

Using the Γ-algebra map $i: S^2\Lambda \to \Lambda \otimes \Lambda$ we can compute $\pi(v(\alpha)\cdot\sigma(\bar{\beta}))=0$. Also, since $\omega \in I_2$, $\gamma_2\sigma(\bar{\beta})=0$. We are thus left with computing $\gamma_2 v(\alpha)$. Choose $z \in \Lambda\otimes\Lambda$ such that it maps to α under $\Lambda\otimes\Lambda \to E_2\Lambda$ and let $\bar{\alpha}$ be its image in $S_2\Lambda$. Then in $\Lambda\otimes\Lambda$

$$iv(\alpha) = (1+T)z$$

so that a computation using (2.19) gives us

$$\begin{aligned} i\gamma_2 v(\alpha) = \gamma_2 iv(\alpha) &= (1+T)\gamma_2 z + z\cdot Tz \\ &= iv(y) + i\sigma(\mu(\bar{\alpha})) \end{aligned}$$

for some $y \in E_2\Lambda$ (in fact y is the image of $\gamma_2 z$). □

We end this section by taking a closer look at the map ϕ. Let V be a module. Then we have

Generators of S_2S^2V:

$$[x,y]\cdot[z,w],$$

$$\sigma(x)\cdot[y,z],$$

$$\sigma(x)\cdot\sigma(y)$$

for any $x,y,z,w \in V$.

Generators of S^2S_2V:

$$[x\cdot y,\ z\cdot w],$$

$$\sigma(x\cdot y)$$

for any $x,y,z,w\in V$.

Here σ is the map of (2.6).

The effect of

$$\phi: S_2S^2V \to S^2S_2V$$

is given by

$$\begin{aligned} &[x,y]\cdot[z,w] \to [x\cdot z, y\cdot w] + [x\cdot w, y\cdot z], \\ &\sigma(x)\cdot[y,z] \to [x\cdot y, x\cdot z], \\ &\sigma(x)\cdot\sigma(y) \to \sigma(x\cdot y). \end{aligned}$$

We can use this to compute the kernel and cokernel of ϕ. First, we have a map

$$\varrho: V^{\otimes 4} \to S_2S^2V$$

given by

$$a\otimes b\otimes c\otimes d \to [a,b]\cdot[c,d] + [a,c]\cdot[b,d] + [a,d]\cdot[b,c].$$

It is easy to see that

$$\phi\varrho = 0.$$

Further, we have a factorization

$$\begin{tikzcd} V^{\otimes 4} \arrow[rr, "\varrho"] \arrow[dr] & & S_2S^2V \\ & E_4V \arrow[ur, "\vartheta"'] & \end{tikzcd}$$

Here E_4V is the fourth exterior power of V i.e. the cokernel of the composite

$$(\Phi V) \otimes V^{\otimes 2} \xrightarrow{d \otimes 1} V^{\otimes 4} \to S_4V$$

where d is from (2.6).

Claim. *The induced map*

$$\overline{\vartheta} : E_4V \to \ker \phi$$

is a linear isomorphism.

Proof. By naturality of ϑ and simplicity of the functor E_4, $\overline{\vartheta}$ is injective, since it is nontrivial. To see surjectivity, we note that $\overline{\vartheta}$ is onto when $\dim V \leq 4$. Thus, since E_4 is a polynomial functor of degree ≤ 4 the result follows. □

Now, an easy calculation shows

$$(S_2V)^* = S^2V^*$$

and

$$(S^2V)^* = S_2V^*.$$

From this and Lemma 2.15 we have

$$\phi^* = \phi.$$

Further $(E_4V)^* = E_4V^*$ so that the claim gives us an exact sequence

$$0 \to E_4V \to S_2S^2V \xrightarrow{\phi} S^2S_2V \to E_4V \to 0$$

which is natural as functors of modules. This defines a map

$$\mathbb{F}_2 \to \mathrm{Ext}^2_{\mathscr{F}}(E_4, E_4),$$

where $\mathscr{F}$ is the category of endofunctors on the category of modules. L. Schwartz has shown (private communication) that this map is an injection.

3. Some explicit computations of homotopy groups

The goal of this section is review the explicit description one can give $\pi_* S_2 V$ and $\pi_* S^2 V$ in terms of $\pi_* V$. From this one can determine all the primary operations for the homotopy of simplicial (co)commutative (co)algebras.

We begin by summarizing the Eilenberg–Zilber theorem as given in [15, 9].

Theorem 3.1. *Let V and W be two simplicial $\mathbb{F}_2$-modules. Then there exists a unique natural chain map*

$$D : N(V) \otimes N(W) \to N(V \otimes W)$$

which is the identity in dimension 0.

Moreover, there exists a natural chain map

$$E : N(V \otimes W) \to N(V) \otimes N(W)$$

such that

$$ED = 1, \qquad DE \simeq 1.$$

In [9] it was noticed that since D is necessarily the shuffle map (see [15]) then D possesses a symmetry. This symmetry was exploited by Dwyer to construct higher order versions of D which we now describe.

Definition 3.2. For each $k \geq 0$, let

$$\phi_k : N(V) \otimes N(W) \to N(V \otimes W)$$

be the chain map such that for $x \in N(V)$ and $y \in N(W)$

$$\phi_k(x \otimes y) = \begin{cases} x \otimes y, & |x| = k = |y|, \\ 0 & \text{otherwise.} \end{cases}$$

ϕ_k is called an admissible map.

Let T denote the switching map for either

$$N(V) \otimes N(W) \to N(W) \otimes N(V)$$

or

$$N(V \otimes W) \to N(W \otimes V).$$

Theorem 3.3. *Let V and W be simplicial $\mathbb{F}_2$-modules. For each $k \geq 0$ there exists a natural chain map*

$$D^k : [N(V) \otimes N(W)]_m \to N(V \otimes W)_{m-k}$$

defined for $m \geq 2k$ and satisfying

1. $D^0 + TD^0T + \phi_0 = D$,
2. $D^{k+1} + TD^{k+1}T + \phi_{k+1} = \partial D^k + D^k\partial$.

Remark. Dwyer showed in [9] that each D^k is unique in a certain sense.

In light of (2.10), a computation of the homotopy of S_2V, for a simplicial module V, in terms of π_*V would give a complete picture of the primary operator algebra for the homotopy of a simplicial commutative algebra. Such a description is known to exist by [8]. We now proceed to make this description explicit.

Fix a simplicial module V. For each $0 \le i \le n$ define

$$\Theta_i : N_nV \to N_{n+i}S_2V \tag{3.4}$$

by

$$\Theta_i(a) = \rho D^{n-i}(a \otimes a) + \rho D^{n-i-1}(a \otimes \partial a) \tag{3.5}$$

where the D^s are from Theorem 3.3.

A computation gives us that

$$\partial\Theta_i = \Theta_i\partial.$$

Thus, for $2 \le i \le n$, Θ_i induces a natural map

$$\bar{\delta}_i : \pi_nV \to \pi_{n+i}S_2V. \tag{3.6}$$

Also, the chain map

$$\rho D : N_sV \otimes N_tV \to N_{s+t}S_2V$$

induces a homomorphism

$$\overline{m} : \pi_sV \otimes \pi_tV \to \pi_{s+t}S_2V. \tag{3.7}$$

Combining the results of [4, 2, 9] we are led to

Proposition 3.8. *Let V be a fixed simplicial module. Define W to be the graded module with basis*

$$\begin{array}{ll} \delta_i(x) & \textit{for } x \in \pi_nV \textit{ and } 2 \le i \le n, \\ x \cdot y & \textit{for } x \in \pi_sV \textit{ and } y \in \pi_tV. \end{array}$$

Define a submodule B in W with basis

$$\delta_i(x+y) + \delta_i(x) + \delta_i(y) + \begin{cases} 0, & 2 \le i < n \quad \textit{for } x, y \in \pi_nV \\ x \cdot y, & i = n, \end{cases}$$

$$\begin{array}{ll} x \cdot y + y \cdot x & \textit{for } x \in \pi_sV \textit{ and } y \in \pi_tV, \\ x \cdot (y+z) + x \cdot y + x \cdot z & \textit{for } x \in \pi_sV \textit{ and } y,z \in \pi_tV, \\ x \cdot x & \textit{for } x \in \pi_nV \textit{ and } n > 0. \end{array}$$

Then the map $W \to \pi_* S_2 V$ *given by*

$$x \cdot y \to \overline{m}(x \otimes y),$$
$$\delta_i x \to \bar{\bar{\delta}}_i x$$

is natural and induces a linear isomorphism

$$W/B \simeq \pi_* S_2 V.$$

Next, consider the composite

$$N_n V \xrightarrow{\Theta_i} N_{n+i} S_2 V \xrightarrow{N_*} N_{n+i} S^2 V \tag{3.9}$$

of chain maps. Here N is the norm map (2.7). This induces a natural map

$$\sigma_i : \pi_n V \to \pi_{n+i} S^2 V \tag{3.10}$$

for each $0 \le i \le n$. Also, the composite

$$N_s V \otimes N_t V \xrightarrow{D} N_{s+t}(V \otimes V) \xrightarrow{\pi_*} N_{s+t} S_2 V \xrightarrow{N_*} N_{s+t} S^2 V \tag{3.11}$$

induces the homomorphism

$$\tau : \pi_s V \otimes \pi_t V \to \pi_{s+t} S^2 V. \tag{3.12}$$

The following is given in [12].

Proposition 3.13. *Let* V *be a simplicial module. Let* T *be the graded vector space with basis*

$$\sigma_i(x) \quad \textit{for } x \in \pi_n V \textit{ and } 0 \le i \le n,$$
$$[x, y] \quad \textit{for } x \in \pi_n V \textit{ and } y \in \pi_m V, \quad n, m \ge 0.$$

Let R *be the submodule of* T *with basis*

$$\begin{array}{ll} [x, y] + [y, x] & \textit{for } x \in \pi_n V, \quad y \in \pi_m V, \quad n, m \ge 0, \\ [x, y+z] + [x, y] + [x, z] & \textit{for } x \in \pi_n V, \quad y, z \in \pi_m V, \quad n, m \ge 0, \end{array}$$

$$\sigma_i(x+y) + \sigma_i(x) + \sigma_i(y) + \begin{cases} 0, & 0 \le i < n, \\ [x, y], & i = n \quad \textit{for } x, y \in \pi_n V, \end{cases}$$

$$[x, x] \quad \textit{for } x \in \pi_n V, \ n \ge 0.$$

Then the map $T \to \pi_* S^2 V$ defined by

$$\sigma_i(x) \to \sigma_i(x),$$
$$[x, y] \to \tau(x \otimes y)$$

induces a natural linear isomorphism

$$T/R \simeq \pi_* S^2 V.$$

Moreover, if we let

$$e : \pi_* S^2 V \to \pi_* V \otimes \pi_* V$$

be induced by the composition of chain maps

$$NS^2 V \xrightarrow{i_*} N(V \otimes V) \xrightarrow{E} NV \otimes NV$$

(see Theorem 3.1) then for $x \in \pi_n V \quad y \in \pi_m V \quad n, m \geq 0$

$$e([x, y]) = x \otimes y + y \otimes x$$

and for $x \in \pi_n V \quad 0 \leq i \leq n$

$$e(\sigma_i(x)) = \begin{cases} 0, & 0 \leq i < n, \\ x \otimes x, & i = n. \end{cases}$$

We take a moment to note a corollary given in [12].

Corollary 3.14. *The effect of the homomorphism*

$$N_* : \pi_* S_2 V \to \pi_* S^2 V$$

is given by

$$x \cdot y \to [x, y]$$

for $x \in \pi_n V$, $y \in \pi_m V$, $n, m \geq 0$, *and*

$$\delta_i(x) \to \sigma_i(x)$$

for $x \in \pi_n V$, $2 \leq i \leq n$. *Moreover, under the homomorphism (2.6)*

$$\iota_* : \Phi \pi_* V \to \pi_* S_2 V$$

we have

$$\operatorname{im} \iota_* = \ker N_*.$$

Finally, given Π in $s\mathscr{C}\mathscr{A}$, then for $x \in \pi_n \Pi$ Proposition 3.13 tells us that

$$\psi_* x = \Sigma[x', x''] + \Sigma \sigma_i(x Sq^i) \tag{3.15}$$

which defines the action of the Steenrod operations. From this and Corollary 3.14 we conclude 1 of (1.5). Also, we define the coproduct

$$\Delta : \pi_* \Pi \to \pi_* \Pi \otimes \pi_* \Pi \tag{3.16}$$

by $e\psi_*$ from Proposition 3.13.

4. Reduction in the proof of the main theorem

In this section, we use the results of the previous two sections to state a theorem which allows us to prove Theorem 1.11. First, recall that if Λ is a simplicial commutative algebra then $S^2\Lambda$ is also a simplicial commutative algebra using (2.16). Further if Λ is a simplicial abelian Hopf algebra then, by Corollary 2.18, the coproduct induces a simplicial algebra map $\psi : \Lambda \to S^2\Lambda$. Thus the induced map ψ_* on homotopy groups is a map of D-algebras. Hence one approach to proving Theorem 1.11 is to determine $\pi_* S^2\Lambda$ as a D-algebra based upon Proposition 3.13. For this we now state

Theorem 4.1. *Let Λ be a simplicial commutative algebra. Then for the associated simplicial commutative algebra $S^2\Lambda$ the following relations hold in the D-algebra $\pi_* S^2\Lambda$:*

(a) *For $x \in \pi_n\Lambda$, $0 \le i \le n$, $2 \le j \le n+i$,*

$$\delta_j\sigma_i(x) = \sum_{j<2s} \binom{s-i-1}{2s-j-1} \sigma_{i+j-s}\delta_s(x).$$

(b) *For $x \in \pi_n\Lambda$, $y \in \pi_m\Lambda$, $2 \le j \le n+m$,*

$$\delta_j[x,y] = \begin{cases} \sigma_j(x\cdot y) + [x\cdot x, \delta_j y] & \text{if } n=0, \\ \sigma_j(x\cdot y) + [\delta_j x, y\cdot y] & \text{if } m=0, \\ \sigma_j(x\cdot y) & \text{otherwise.} \end{cases}$$

(c) *For $x \in \pi_n\Lambda$, $y \in \pi_m\Lambda$, $0 \le i \le n$, $0 \le j \le m$*

$$\sigma_i(x)\cdot\sigma_j(y) = \sigma_{i+j}(x\cdot y).$$

(d) *For $x \in \pi_n\Lambda$, $y,z \in \pi_*\Lambda$, $0 \le i \le n$*

$$\sigma_i(x)\cdot[y,z] = \begin{cases} [x\cdot y, x\cdot z] & \text{if } i=n, \\ 0 & \text{otherwise.} \end{cases}$$

(e) *For $x,y,z,w \in \pi_*\Lambda$*

$$[x,y]\cdot[z,w] = [x\cdot z, y\cdot w] + [x\cdot w, y\cdot z].$$

With this we now pause to prove Theorem 1.11. First, by Theorems 1.3 and 1.6, π_*H is both a D-algebra and an A-coalgebra. Moreover, Δ is a map of simplicial commutative algebras by (2.10) and Lemma 2.14. By Theorems 3.1 and 1.3 we conclude π_*H is a Hopf Γ-algebra.

We now proceed to establish 1 and 2 of Definition 1.10. For the remainder of this section we fix $x \in \pi_n H$ and write

$$\psi_* x = \sum_k [x'_k, x''_k] + \sum_s \sigma_s(xSq^s)$$

as in (3.15).

1. The first part is an easy consequence of the fact that Δ is a map of simplicial commutative algebras. For the second part let $y \in \pi_m H$ and write

$$\psi_* y = \sum_l [y_l', y_l''] + \sum_t \sigma_t(xSq^t).$$

By Theorem 4.1 we have

$$\begin{aligned}(\psi_* x)\cdot(\psi_* y) &\equiv \sum_{s,t} \sigma_s(xSq^s)\cdot\sigma_t(ySq^t)\\ &\equiv \sum_{i\geq 0}\sum_{s+t=i} \sigma_i(xSq^s\cdot ySq^t)\\ &\equiv \sum_{i\geq 0}\sigma_i\left(\sum_{s+t=i} xSq^s\cdot ySq^t\right),\end{aligned}$$

where, herein after "$\equiv$" means "equal modulo [,]'s". By (3.15) we have

$$\psi_*(x\cdot y) \equiv \sum_{i\geq 0}\sigma_i((x\cdot y)Sq^i).$$

The conclusion follows from Corollary 2.17. □

2. Fix $2 \leq j < n$. By (3.15) we have

$$\psi_*\delta_j(x) \equiv \sum_i \sigma_i((\delta_j x)Sq^i).$$

Next, Theorem 4.1 gives us

$$\begin{aligned}\delta_j\psi_* x &\equiv \sum_k \sigma_j(x_k'\cdot x_k'') + \sum_s \delta_j\sigma_s(xSq^s)\\ &\equiv \sum_k \sigma_j(x_k'\cdot x_k'') + \sum_s\sum_{j<2l}\binom{l-s-1}{2l-j-1}\sigma_{j+s-l}\delta_l(xSq^s)\\ &\equiv \sum_k \sigma_j(x_k'\cdot x_k'') + \sum_s\sum_{2i-j<2s}\binom{j-i-1}{j-2i+2s-1}\sigma_i\delta_{j-i+s}(xSq^s)\\ &\equiv \sum_k \sigma_j(x_k'\cdot x_k'') + \sum_i\sigma_i\left(\sum_{2i-j<2s}\binom{j-i-1}{j-2i+2s-1}\delta_{j-i+s}(xSq^s)\right)\end{aligned}$$

When $i<j$ we immediately get the third equation. When $i>j$ the expression

$$\binom{m}{r} = \binom{-m+r-1}{r}$$

gives us the first equation. When $i=j$ we just need to verify

$$\xi\gamma_2 x = \sum x_k'\cdot x_k''$$

which is just a consequence of Proposition 2.22. Finally, combining Theorem 4.1 and Definition 1.1 we get

$$\begin{aligned}\delta_n\psi_* x &\equiv \sum_k \sigma_n(x_k'\cdot x_k'') + \sum_s \delta_n\sigma_s(xSq^s) + \sum_{s<t}\sigma_s(xSq^s)\cdot\sigma_t(xSq^t)\\ &\equiv \sum_k \sigma_n(x_k'\cdot x_k'') + \sum_s \delta_n\sigma_s(xSq^s) + \sum_{i\geq 0}\sum_{2s<i}\sigma_i(xSq^s\cdot xSq^{i-s})\\ &\equiv \sum_k \sigma_n(x_k'\cdot x_k'') + \sum_s \delta_n\sigma_s(xSq^s) + \sum_{i\geq 0}\sigma_i\left(\sum_{2s<i} xSq^s\cdot xSq^{i-s}\right)\end{aligned}$$

and so proceeding as before gives us the remaining equations. The conclusion follows from Corollary 2.17. This completes the proof of Theorem 1.11. □

We end this section by describing our strategy for proving Theorem 4.1. Our main objective is to reduce our computations to ones in group cohomology. For this we want to follow the classic approach developed by Adem in [1] as made systematic by Dwyer in [9]. To accomplish this we first note that Theorem 4.1 easily follows from:

Proposition 4.2. *Let V be a simplicial module. Then the effect of the map*

$$\phi : S_2S^2V \to S^2S_2V$$

in homotopy is given by the following:

(a) *For $x\in\pi_nV, 0\leq i\leq n, 2\leq j\leq n+i$*

$$\phi_*\delta_j\sigma_i(x) = \sum_{j<2s}\binom{s-i-1}{2s-j-1}\sigma_{i+j-s}\delta_s(x).$$

(b) *For $x\in\pi_nV, y\in\pi_mV,\ 2\leq i\leq n+m$,*

$$\phi_*\delta_i[x,y] = \begin{cases}\sigma_i(x\cdot y) + [x\cdot x,\delta_i x] & \text{for } n=0,\\ \sigma_i(x\cdot y) + [\delta_i x, y\cdot y] & \text{for } m=0,\\ \sigma_i(x\cdot y) & \text{otherwise.}\end{cases}$$

(c) *For $x\in\pi_nV,\ y\in\pi_mV,\ 0\leq i\leq n,\ 0\leq j\leq m$,*

$$\phi_*(\sigma_i(x)\cdot\sigma_j(y)) = \sigma_{i+j}(x\cdot y).$$

(d) *For $x\in\pi_nV,\ y,z\in\pi_*V,\ 0\leq i\leq n$,*

$$\phi_*(\sigma_i(x)\cdot[y,z]) = \begin{cases}[x\cdot y, x\cdot z], & i=n,\\ 0 & \text{otherwise.}\end{cases}$$

(e) *For $x,y,z,w\in\pi_*V$,*

$$\phi_*([x,y]\cdot[z,w]) = [x\cdot z, y\cdot w] + [x\cdot w, y\cdot z].$$

We are now reduced to analyzing the natural map $\phi : S_2S^2 \to S^2S_2$ of functors on simplicial vector spaces as induced in homotopy. Unfortunately, this map cannot directly be described in a way that will allow us to use Dwyer's techniques for translating to group cohomology. To get around this, in the next section we use the normalizing map $N : S_2 \to S^2$ to show that ϕ fits into a commuting diagram involving a natural map $\alpha : S^2S^2 \to S^2S^2$. Using the fact that $S^2S^2V = (V^{\otimes 4})^{D_8}$, where $D_8 < \Sigma_4$ is the dihedral group of order 8, we show that the map α does have the properties that allows us to use Dwyer's techniques and, in fact, easily translate to map of group cohomology that is computable. In Section 6, we briefly review the techniques of group cohomology that we need which will be used in the final section to make our required calculations.

5. Further reductions

In this section, we show how translate information from the map $\phi : S_2S^2V \to S^2S_2V$ to a new map $\alpha : S^2S^2V \to S^2S^2V$ whose properties make it much more useful for calculations.

We start by recalling from Section 3, that we have the norm map

$$N_V : S_2V \to S^2V$$

whose effect is

$$x \cdot y \to [x, y].$$

Consider now the maps

$$N_{S^2V} : S_2S^2V \to S^2S^2V$$

and

$$S^2N_V : S^2S_2V \to S^2S_2V.$$

It is well known that $S^2S^2V = S^{\Sigma_2 \int \Sigma_2}V$ where $\Sigma_2 \int \Sigma_2$ is the wreath product of Σ_2 and Σ_2, i.e. the subgroup of Σ_4 which fits into the split extension

$$1 \to \Sigma_2 \times \Sigma_2 \to \Sigma_2 \int \Sigma_2 \to \Sigma_2 \to 1 \tag{5.1}$$

where, in terms of transpositions, we have

$$\Sigma_2 \times \Sigma_2 = \langle (1,2), (3,4) \rangle,$$

$$\Sigma_2 = \langle (1,3)(2,4) \rangle.$$

Moreover, it is well-known that $\Sigma_2 \int \Sigma_2 \simeq D_8$; the dihedral group of order 8. We thus have the identity

$$S^2S^2V \simeq S^{D_8}V. \tag{5.2}$$

Lemma 5.3. *There exists a natural idempotent map*

$$\alpha : S^{D_8}V \to S^{D_8}V$$

such that the diagram

$$\begin{tikzcd} S_2S^2V \arrow[r, "N_{S^2V}"] \arrow[d, "\phi"'] & S^{D_8}V \arrow[d, "\alpha"] \\ S^2S_2V \arrow[r, "S^2N_V"'] & S^{D_8}V \end{tikzcd}$$

commutes. Explicitly

$$\alpha = 1 + r(\Sigma_4, D_8)t(D_8, \Sigma_4).$$

The proof will follow from the next lemma.

Lemma 5.4. *There exists a natural map*

$$\alpha'' : S^2(V^{\otimes 2}) \to (S^2V)^{\otimes 2}$$

such that the diagram

$$\begin{tikzcd} S_2(V^{\otimes 2}) \arrow[r, "N_{V^{\otimes 2}}"] \arrow[d, "\phi''"'] & S^2(V^{\otimes 2}) \arrow[d, "\alpha''"] \\ (S_2V)^{\otimes 2} \arrow[r, "(N_V)^{\otimes 2}"'] & (S^2V)^{\otimes 2} \end{tikzcd}$$

commutes. Here ϕ'' is the map of Lemma 2.14. Indeed, we can take

$$\alpha'' = \varepsilon t(\Sigma_2, \Sigma_2 \times \Sigma_2)$$

where the transfer is associated with the diagonal $\Sigma_2 \to \Sigma_2 \times \Sigma_2$ and ε is the isomorphism induced by $1 \otimes T \otimes 1 : V^{\otimes 4} \to V^{\otimes 4}$.

Proof. First, we have commuting diagrams

$$\begin{tikzcd} V^{\otimes 4} \arrow[r, "1+(1,3)(2,4)"] \arrow[d, "p_{V^{\otimes 2}}"'] & V^{\otimes 4} \arrow[d, "i_{V^{\otimes 2}}"] \\ S_2(V^{\otimes 2}) \arrow[r, "N_{V^{\otimes 2}}"'] & S^2(V^{\otimes 2}) \end{tikzcd}$$

and

$$\begin{tikzcd} V^{\otimes 4} \arrow[r, "(1+(1,2))\cdot(1+(3,4))"] \arrow[d, "\rho_V^{\otimes 2}"'] & V^{\otimes 4} \arrow[d, "i_V^{\otimes 2}"] \\ (S_2 V)^{\otimes 2} \arrow[r, "N_V^{\otimes 2}"'] & (S^2 V)^{\otimes 2}. \end{tikzcd}$$

An easy computation shows that the diagram

$$\begin{tikzcd} V^{\otimes 4} \arrow[r, "1+(1,3)(2,4)"] \arrow[d, "(2,3)"'] & V^{\otimes 4} \arrow[d, "(2,3)(1+(1,3))"] \\ V^{\otimes 4} \arrow[r, "(1+(1,2))(1+(3,4))"] & V^{\otimes 4} \end{tikzcd}$$

commutes. Consider now the map

$$S^2(V^{\otimes 2}) \xrightarrow{i_{V^{\otimes 2}}} V^{\otimes 4}.$$

In the group ring $\mathbb{F}_2[\Sigma_4]$, we have the identity

$$\begin{aligned}(2,3)(1+(1,3))(1,3)(2,4) &= (1+(1,2))(2,3)(1,3)(2,4) \\ &= (1+(1,2))(1,2)(3,4)(2,3) \\ &= ((1,2)(3,4)+(3,4))(2,3).\end{aligned}$$

This shows that the image of the above map is invariant under the action of $\langle(1,2),(3,4)\rangle$. We thus have a commuting diagram

$$\begin{tikzcd} S_2(V^{\otimes 2}) \arrow[r, "i_{V^{\otimes 2}}"] \arrow[d, "\alpha''"'] & V^{\otimes 4} \arrow[d, "(2,3)(1+(1,3))"] \\ (S_2 V)^{\otimes 2} \arrow[r, "i_V^{\otimes 2}"'] & V^{\otimes 4} \end{tikzcd}$$

defining α''.

Combining these four diagrams and Lemma 2.14 gives us a cube

$$\begin{tikzcd}
V^{\otimes 4} \arrow[rr] \arrow[dd] \arrow[dr] & & V^{\otimes 4} \arrow[dd] & \\
& S_2(V^{\otimes 2}) \arrow[rr] \arrow[dd] & & S^2(V^{\otimes 2}) \arrow[ul] \arrow[dd] \\
V^{\otimes 4} \arrow[rr] \arrow[dr] & & V^{\otimes 4} & \\
& (S_2V)^{\otimes 2} \arrow[rr] & & (S^2V)^{\otimes 2} \arrow[ul]
\end{tikzcd}$$

from which our desired commutative diagram results. The identification of α'' follows from our construction and the definition of transfer. □

Proof of Lemma 5.3. Consider the composite

$$S^2S^2V \xrightarrow{S^2(i_V)} S^2(V^{\otimes 2}) \xrightarrow{\alpha''} (S^2V)^{\otimes 2}.$$

From Lemma 5.4 and a computation we have

$$\alpha''(1,2)(3,4) = \varepsilon(1,2)(3,4)t(\Sigma_2, \Sigma_2 \times \Sigma_2) = (1,3)(2,4)\alpha''.$$

Thus $(1,3)(2,4)\alpha'' S^2(i_V) = \alpha'' S^2(i_V)$. Hence, we have a diagram

$$\begin{tikzcd}
S^2S^2(V) \arrow[r, "S^2(i_V)"] \arrow[d, "\alpha"'] & S^2(V^{\otimes 2}) \arrow[d, "\alpha''"] \\
S^2S^2(V) \arrow[r, "i_{S^2V}"'] & (S^2V)^{\otimes 2}.
\end{tikzcd}$$

By Lemmas 2.15 and 5.4 our desired diagram commutes. From this and the identity $(2,3)(1,3) = (1,3)(1,2)$ we arrive at the commuting diagram

$$\begin{tikzcd}
S^2S^2(V) \arrow[r] \arrow[d, "\alpha"'] & V^{\otimes 4} \arrow[d, "(1,3)+(2,3)"] \\
S^2S^2(V) \arrow[r] & V^{\otimes 4}
\end{tikzcd}$$

Clearly $1, (2,3), (1,3)$ are coset representatives for D_8 in Σ_4. Also $((2,3)+(1,3))^2 = (1,3)(1,2)+(2,3)(1,2)$ from the above and the identity $(1,3)(2,3) = (2,3)(1,2)$. Hence $\alpha^2 = \alpha$. □

Corollary 5.5. *The following cube commutes:*

$$\begin{tikzcd}
S_2S^2V \arrow[rr] \arrow[dr] \arrow[dd, "\phi"'] & & S^2S^2V \arrow[dr] \arrow[dd, "\alpha"] & \\
& S_2(V^{\otimes 2}) \arrow[rr] \arrow[dd, "\phi''"] & & S^2(V^{\otimes 2}) \arrow[dd, "\alpha''"] \\
S^2S_2V \arrow[rr] \arrow[dr] & & S^2S^2V \arrow[dr] & \\
& (S_2V)^{\otimes 2} \arrow[rr] & & (S^2V)^{\otimes 2}
\end{tikzcd}$$

Proof. This easily follows from Lemmas 2.15, 5.3, 5.4, and naturality. □

Note. The effect of the map

$$\alpha : S^{D_8}V \to S^{D_8}V$$

on elements is

$$\begin{aligned}
&[[x,y],[z,w]] \to [[x,z],[y,w]] + [[x,w],[y,z]],\\
&[\sigma(x),[y,z]] \to [[x,y],[x,z]],\\
&[\sigma(x),\sigma(y)] \to \sigma[x,y],\\
&\sigma[x,y] \to \sigma[x,y],
\end{aligned}$$

from which we easily verify idempotence. We further note that the module of natural maps

$$(-)^{D_8} \to (-)^{D_8}$$

on the category of Σ_4-modules has as basis the set $\{1,\alpha\}$. In light of this, Lemma 5.3 should not be surprising.

Now, by Proposition 3.13 we have

Proposition 5.6. *The following are generators of* $\pi_*S^2S^2V$:

(a) $\sigma_j\sigma_i(x)$ *for* $x \in \pi_nV$, $0 \le i \le n$, $0 \le j \le n+i$.
(b) $\sigma_i[x,y]$ *for* $x \in \pi_nV$, $y \in \pi_mV$, $0 \le i \le n+m$.
(c) $[\sigma_i(x),\sigma_j(y)]$ *for* $x \in \pi_nV$, $y \in \pi_mV$, $0 \le i \le n$, $0 \le j \le m$.
(d) $[\sigma_i(x),[y,z]]$ *for* $x \in \pi_nV$, $y,z \in \pi_*V$, $0 \le i \le n$.
(e) $[[x,y],[z,w]]$ *for* $x,y,z,w \in \pi_*V$.

By Corollary 3.14, the effect of the map

$$(N_{S^2V})_* : \pi_*S_2S^2V \to \pi_*S^2S_2V$$

is given by

$$\delta_j\sigma_i(x) \to \sigma_j\sigma_i(x),$$
$$\delta_j[x,y] \to \sigma_j[x,y],$$
$$\sigma_i(x)\cdot\sigma_j(y) \to [\sigma_i(x),\sigma_j(y)],$$
$$\sigma_i(x)\cdot[y,z] \to [\sigma_i(x),[y,z]],$$
$$[x,y]\cdot[z,w] \to [[x,y],[z,w]].$$

Also, the effect of the map

$$(S^2N_V)_* : \pi_*S^2S_2V \to \pi_*S^2S^2V$$

is given by

$$\sigma_i\delta_j(x) \to \sigma_i\sigma_j(x),$$
$$\sigma_i(x\cdot y) \to \sigma_i[x,y],$$
$$[\delta_i(x),\delta_j(y)] \to [\sigma_i(x),\sigma_j(y)],$$
$$[\delta_i(x), y\cdot z] \to [\sigma_i(x),[y,z]],$$
$$[x\cdot y, z\cdot w] \to [[x,y],[z,w]].$$

Further, by Proposition 3.13, the effect of the map

$$(S_2i_V)_* : \pi_*S_2S^2V \to \pi_*S_2(V^{\otimes 2})$$

is given by

$$\delta_j\sigma_i(x) \to \begin{cases} \delta_j(x\otimes x), & i=|x|, \\ 0 & \text{otherwise,} \end{cases}$$

$$\delta_j[x,y] \to \delta_j(x\otimes y + y\otimes x),$$

$$\sigma_i(x)\cdot\sigma_j(y) \to \begin{cases} (x\otimes x)\cdot(y\otimes y), & i=|x|,\ j=|y|, \\ 0 & \text{otherwise,} \end{cases}$$

$$\sigma_i(x)\cdot[y,z] \to \begin{cases} (x\otimes x)\cdot(y\otimes z + z\otimes y), & i=|x|, \\ 0 & \text{otherwise,} \end{cases}$$

$$[x,y]\cdot[z,w] \to (x\otimes y + y\otimes x)\cdot(z\otimes w + w\otimes z).$$

Also, the effect of the map

$$(i_{S_2V})_* : \pi_*S^2S_2V \to \pi_*(S_2V)^{\otimes 2}$$

is given by

$$\sigma_i\delta_j(x) \to \begin{cases} \delta_j(x)\otimes\delta_j(x), & i=|x|+j, \\ 0 & \text{otherwise,} \end{cases}$$

$$\sigma_i(x \cdot y) \to \begin{cases} (x \cdot y) \otimes (x \cdot y), & i = |x| + |y|, \\ 0 & \text{otherwise,} \end{cases}$$

$$[\delta_i(x), \delta_j(y)] \to \delta_i(x) \otimes \delta_j(y) + \delta_j(x) \otimes \delta_i(x),$$
$$[\delta_i(x), y \cdot z] \to \delta_i(x) \otimes (y \cdot z) + (y \cdot z) \otimes \delta_i(x),$$
$$[x \cdot y, z \cdot w] \to (x \cdot y) \otimes (z \cdot w) + (z \cdot w) \otimes (x \cdot y).$$

From this we conclude that the map

$$(S^2 N_V)_* \oplus (i_{S_2 V})_* : \pi_* S^2 S_2 V \to \pi_* S^2 S^2 V \oplus \pi_*(S_2 V)^{\oplus 2}$$

is injective. We are thus reduced, by Corollary 5.5, to computing, in homotopy, the maps induced by α and ϕ''. For this we have

Proposition 5.7. *Let V be a simplicial module. Then the effect of*

$$\alpha_* : \pi_* S^2 S^2 V \to \pi_* S^2 S^2 V$$

is given by

(a) *For $x \in \pi_n V$, $0 \le i \le n$, $0 \le j \le n+1$,*

$$\alpha_* \sigma_j \sigma_i(x) = \sum_{j < 2s} \binom{s-i-1}{2s-j-1} \sigma_{i+j-s} \sigma_s(x).$$

(b) *For $x \in \pi_n V$, $y \in \pi_m V$, $0 \le i \le n+m$,*

$$\alpha_* \sigma_i [x, y] = \sigma_i [x, y].$$

(c) *For $x \in \pi_n V$, $y \in \pi_m V$, $0 \le i \le n$, $0 \le j \le m$,*

$$\alpha_* [\sigma_i(x), \sigma_j(y)] = \sigma_{i+j}[x, y].$$

(d) *For $x \in \pi_n V$, $y, z \in \pi_* V$, $0 \le i \le n$,*

$$\alpha_* [\sigma_i(x), [y, z]] = \begin{cases} [[x, y], [x, z]], & i = n, \\ 0 & \textit{otherwise.} \end{cases}$$

(e) *For $x, y, z, w \in \pi_* V$,*

$$\alpha_* [[x, y], [z, w]] = [[x, z], [y, w]] + [[x, w], [y, z]].$$

Proposition 5.8. *Let V be a simplicial module. Then the effect of*

$$\phi''_* : \pi_* S_2(V^{\otimes 2}) \to \pi_*(S_2 V)^{\otimes 2}$$

is given by

(a) *For $x \in \pi_n V$, $y \in \pi_n V$, $y \in \pi_m V$, $2 \le j \le n+m$,*

$$\phi''_* \delta_j(x \otimes y) = \begin{cases} \delta_j x \otimes (y \cdot y), & m = 0, \\ (x \cdot x) \otimes \delta_j y, & n = 0, \\ 0 & \textit{otherwise.} \end{cases}$$

(b) *For* $x, y, z, w \in \pi_* V$,

$$\phi''_*[(x \otimes y) \cdot (z \otimes w)] = (x \cdot z) \otimes (y \cdot w).$$

We will actually prove a much more general result then Proposition 5.8. To state it we first need the following set up.

Let V and W be modules. Then the map

$$1 \otimes T \otimes 1 \colon V \otimes W \otimes V \otimes W \to V \otimes V \otimes W \otimes W$$

induces

$$\overline{\phi}'' \colon S_2(V \otimes W) \to S_2 V \otimes S_2 W.$$

Following the proof of Lemma 5.4 verbatim gives us

Lemma 5.9. *There exists a map*

$$\overline{\alpha}'' \colon S^2(V \otimes W) \to S^2 V \otimes S^2 W$$

such that the diagram

$$\begin{tikzcd} S_2(V \otimes W) \arrow[r, "N_{V \otimes W}"] \arrow[d, "\phi''"'] & S^2(V \otimes W) \arrow[d, "\overline{\alpha}''"] \\ S_2 V \otimes S_2 W \arrow[r, "N_V \otimes N_W"'] & S^2 V \otimes S^2 W \end{tikzcd}$$

commutes. Indeed we can take

$$\overline{\alpha}'' = \varepsilon t(\Sigma_2, \Sigma_2 \times \Sigma_2)$$

as in Lemma 5.4.

Proposition 5.10. *Let V and W be simplicial modules. Then the effect of*

$$\overline{\phi}'' \colon \pi_* S_2(V \otimes W) \to \pi_* S_2 V \otimes S_2 W$$

is given by

(a) *For* $x \in \pi_n V$, $y \in \pi_m V$, $2 \le j \le n + m$

$$\overline{\phi}''_* \delta_j(x \otimes y) = \begin{cases} \delta_j x \otimes (y \cdot y), & m = 0, \\ (x \cdot x) \otimes \delta_j y, & n = 0, \\ 0 & \textit{otherwise}. \end{cases}$$

(b) *For* $x, z \in \pi_* V$, $y, w \in \pi_* W$,

$$\overline{\phi}''_*[(x \otimes y) \cdot (z \otimes w)] = (x \cdot z) \otimes (y \cdot w).$$

Clearly Proposition 5.10 implies Proposition 5.8. Finally, Proposition 4.2 follows from Lemmas 5.3 and 5.4, Propositions 5.7 and 5.8. The proof of Propositions 5.7 and 5.10 will be given in the last section.

6. Recollections on group cohomology

In this section, we gather the tools necessary for proving Propositions 5.7 and 5.10. The key is the following theorem found in [9].

Theorem 6.1. *Given a simplicial module V and a subgroup $G \leq \Sigma_m$ there exists a natural homomorphism*

$$\Psi^G : \pi_i S^G V \to \bigoplus_{0 \leq k} H^k(G; \pi_{i+k} V^{\otimes m})$$

such that for a subgroup $H \leq G$

$$r(G,H)\Psi^G = \Psi^H r(G,H), \qquad t(H,G)\Psi^H = \Psi^G t(H,G).$$

The usefulness of the map of Theorem 6.1 is now made precise by the following.

Proposition 6.2. *For any simplicial module V, the natural homomorphism of Theorem 6.1 is injective for the group Σ_2.*

Note. Theorem 6.1 is different from Proposition (5.1) in [9] in that the source involves S^G. One can obtain our map directly from Dwyer's construction by ignoring the introduction of the norm element. Furthermore, Proposition 6.2 is a strengthening of Lemma 5.11 in [9] and, as such, we present its modified proof making liberal reference to section 5 of [9].

Proof. We first prove the result for $V = K(n)$ the Eilenberg–MacLane object. From [9] we have

$$N_s(K(n) \otimes K(n)) = \begin{cases} 0, & s < n, \\ \text{nonzero}, & n \leq s \leq 2n, \\ 0, & 2n < s. \end{cases}$$

Let C be the Σ_2-chain complex such that

$$C_s = \begin{cases} \mathbb{F}_2[\Sigma_2]\langle x_s \rangle, & n < s \leq 2n, \\ \mathbb{F}_2\langle y \rangle, & n = s, \\ 0 & \text{otherwise}. \end{cases}$$

If we write $\Sigma_2 = \{1, T\}$, then the differential ∂ on C is given by

$$\begin{aligned} \partial x_{s+1} &= (1+T)x_s, \quad n < s < 2n, \\ \partial x_{n+1} &= y. \end{aligned}$$

Write $\pi_n\ K(n) = \mathbb{F}_2\langle a\rangle$ and define a map

$$f : C \to N(K(n)\otimes K(n))$$

by

$$x_s \to D^{2n-s}(a\otimes a),$$
$$y \to \phi_n(a\otimes a).$$

By Proposition 3.3 of [9] this is a map of Σ_2-chain complexes. Moreover, it is a quasi-isomorphism. Let F be the functor $H^0(\Sigma_2; -)$. We wish to compute

$$H_nF(C) \to \mathscr{R}^{-n}F(C).$$

To do so define the complex $\widehat{C}$ by

$$\widehat{C}_s = \begin{cases} \mathbb{F}_2[\Sigma_2]\langle \widehat{x}_s\rangle, & s \le 2n, \\ 0, & \text{otherwise} \end{cases}$$

with differential $\widehat{\partial}$ given by $\widehat{\partial}\widehat{x}_{s+1} = (1+T)\widehat{x}_s$. This is clearly a free Σ_2-chain complex and the map

$$C \to \widehat{C}$$

given by

$$x_s \to \widehat{x}_s,$$
$$y \to (1+T)\widehat{x}_n$$

is clearly a quasi-isomorphism. Thus, $\mathscr{R}F(C) = F(\widehat{C})$ and an easy calculation gives that

$$H_sF(C) \to \mathscr{R}^{-s}F(C)$$

is an injection for all s.

To obtain the general case, we first take V so that NV is bounded above. Then we have a weak equivalence

$$\bigoplus_\alpha K(n_\alpha) \to V.$$

Thus it suffices to show that if Ψ^{Σ_2} is injective for W_1 and W_2 then it is injective for $W_1 \oplus W_2$. First, we have a decomposition of

$$N((W_1 \oplus W_2)\otimes(W_1 \oplus W_2))$$

as

$$N(W_1\otimes W_1) \oplus N(W_2 \otimes W_2) \oplus N((W_1\otimes W_2)\oplus(W_2\otimes W_1)).$$

Since the last summand is Σ_2-free and since Ψ^{Σ_2} respects this decomposition, injectivity follows. A limit argument completes the proof. $\square$

Corollary 6.3. *For any simplicial module V, Ψ^G is injective for $G = \Sigma_2 \times \Sigma_2$ and $G = \Sigma_2 \int \Sigma_2$.*

The proof is just a specialized adaptation of Lemmas 5.12 and 5.13 of [9].

We now review some basic tools from the cohomology of groups (see [5] or [11]). Consider the extension of finite groups

$$K \rightarrowtail G \twoheadrightarrow Q.$$

Let M be a G-module. Then we have a first quadrant spectral sequence

$$E_2^{*,*} = H^*(Q; H^*(K; M)) \Rightarrow H^*(G; M). \tag{6.4}$$

Here $H^*(K;M)$ is a Q-module since we have the functor

$$H^0(K;-) : \mathscr{C}_G \to \mathscr{C}_Q.$$

To make this spectral sequence useful we have

Lemma 6.5. *Given a diagram*

$$\begin{array}{ccccc} K & \rightarrowtail & G & \twoheadrightarrow & Q \\ \downarrow & & \downarrow & & \downarrow \\ K' & \rightarrowtail & G' & \twoheadrightarrow & Q' \end{array}$$

whose rows are extensions then the induced map

$$H^*(Q'; H^*(K'; M)) \to H^*(Q; H^*(K; M))$$

is a map of spectral sequences for a Q-module M. Moreover, the induced map on E_∞ is compatible with

$$H^*(G'; M) \to H^*(G; M).$$

Further, if the vertical maps are injective, then the map

$$H^*(Q; H^*(K; M)) \to H^*(Q'; H^*(K'; M))$$

induced from the associated transfers, becomes a map of spectral sequences. Again, the induced map on E^∞ is compatible with the associated transfer,

$$H^*(G; M) \to H^*(G'; M).$$

Let H, K be subgroups of a finite group G. A double coset representation of G with respect to H and K is a subset $S \subset G$ such that

$$G = \bigcup_{\sigma \in S} H\sigma K$$

and is minimal among all such subsets. Next, if $x \in G$ and $J \leq G$ define the conjugation map

$$c_x : J \to xJx^{-1}$$

by $c_x(u) = xux^{-1}$.

Proposition 6.6. *Let S be a double coset representation of G with respect to H and K and let M be a G-module. Then for $\alpha \in H^*(K;M)$*

$$\begin{aligned} r(G,H)t(K,G)(\alpha) &= \sum_{x \in S} t(H \cap xKx^{-1}, H) r(xKx^{-1}, H \cap xKx^{-1}) c_x(\alpha) \\ &= \sum_{x \in S} t(H \cap xKx^{-1}, H) c_x r(K, x^{-1}Hx \cap K)(\alpha) \end{aligned}$$

holds in $H^*(H;M)$.

Finally, we recall (see [5, p.256]) that if $u \in H^*(K;M)$ and $x \in H^*(G;M)$ then

$$t(K,G)[(r(G,K)u) \cdot x] = u \cdot t(K,G)x. \tag{6.7}$$

7. Final proof

In this section, we prove Propositions 5.7 and 5.10 using the methods of the previous section. First, we need some basic results to facilitate our computations.

Let $K(n)$ be the Eilenberg–MacLane module so that $\pi_* K(n) = \mathbb{F}_2\langle a \rangle$ where $|a| = n \geq 0$. Then by Proposition 3.13

$$\pi_* S^2 K(n) \simeq \begin{cases} \mathbb{F}_2\langle \sigma_i(a) \rangle, & * = n + i,\ 0 \leq i \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

Also $H^*(\Sigma_2; \mathbb{F}_2) = \mathbb{F}_2[w]$ where w is dual to the generator $H_1(\Sigma_2; \mathbb{F}_2) \simeq \mathbb{F}_2$. We then have

Proposition 7.1. *Under the homomorphism*

$$\Psi^{\Sigma_2} : \pi_* S^2 K(n) \to H^*(\Sigma_2; \mathbb{F}_2)$$

of Theorem 6.1

$$\Psi^{\Sigma_2} \sigma_i(a) = w^{2n-i}$$

for all $0 \leq i \leq n$.

Proof. This follows easily from Proposition 6.2. □

Now, take $K(m)$ so that $\pi_* K(m) \simeq \mathbb{F}_2\langle b \rangle$ where $|b| = m \geq 0$.

Proposition 7.2. *Let M be the Σ_2-submodule of $\pi_*\big(K(n)\times K(m)\big)^{\otimes 2}$ generated by $a\otimes b$. Then*

$$H^i(\Sigma_2;M)=\begin{cases}0, & i>0,\\ \mathbb{F}_2\langle\Psi^{\Sigma_2}[a,b]\rangle, & i=0.\end{cases}$$

Proof. For $i>0$ this just follows from the fact that M is a free Σ_2-module. For $i=0$ we note that under the projections

$$S^2(K(n)\times K(m))\to S^2K(n),$$
$$S^2(K(n)\times K(m))\to S^2K(m),$$

$[a,b]$ projects to 0 in homotopy. Hence, by naturality and Proposition 6.2 the result follows. $\square$

Proposition 7.3. *Consider the extension*

$$\Sigma_2\times\Sigma_2\rightarrowtail D_8\twoheadrightarrow\Sigma_2.$$

Then for a simplicial module V

$$H^*(D_8;\pi_*V^{\otimes 4})\simeq H^*(\Sigma_2;H^*(\Sigma_2;\pi_*V^{\otimes 2})^{\otimes 2}).$$

Moreover, we have a factorization

$$\begin{tikzcd}
\pi_*S^{D_8}V \arrow[rr, "\psi^{D_8}"] \arrow[d, equal] & & H^*(D_8;\pi_*V^{\otimes 4}) \arrow[dd, equal] \\
\pi_*S^2S^2V \arrow[d, "\psi^{\Sigma_2}"'] & & \\
H^*(\Sigma_2;\pi_*(S^2V)^{\otimes 2}) \arrow[rr, "H^*(\Sigma_2;(\psi^{\Sigma_2})^{\otimes 2})"'] & & H^*(\Sigma_2;H^*(\Sigma_2;\pi_*V^{\otimes 2})^{\otimes 2})
\end{tikzcd}$$

Proof. Define functors

$$F_1:\mathscr{C}_{D_8}\to\mathscr{C}_{\Sigma_2}$$

and

$$F_2:\mathscr{C}_{\Sigma_2}\to\text{(modules)}$$

by $F_1=H^0(\Sigma_2\times\Sigma_2;-)$ and $F_2=H^0(\Sigma_2;-)$. Then F_1 preserves injectives and $F_2\circ F_1=H^0(D_8;-)$. So by (5.8) of [9],

$$\mathscr{R}(F_2\circ F_1)\simeq\mathscr{R}F_2\circ\mathscr{R}F_1.$$

Thus, it suffices to compute $H_*(\mathscr{R}F_2 \circ \mathscr{R}F_1)$ for $NV^{\otimes 4}$. Since we have an equivariant equivalence

$$NV^{\otimes 4} \to (\pi_* V)^{\otimes 4}$$

and since $\mathscr{R}F_1$ is Σ_2-equivalent to $\mathscr{R}F_2 \otimes \mathscr{R}F_2$ we conclude that we have a Σ_2-equivalence

$$\mathscr{R}F_1(NV^{\otimes 4}) \to H^*(\Sigma_2; \pi_* V^{\otimes 2})^{\otimes 2}$$

so by (5.10) of [9]

$$H_*((\mathscr{R}F_2 \circ \mathscr{R}F_1)(NV^{\otimes 4})) \simeq H^*(\Sigma_2; H^*(\Sigma_2; \pi_* V^{\otimes 2})^{\otimes 2}).$$

The identification of Ψ^{D_8} follows, again, from (5.8) of [9]. □

Note. The identification in Proposition 7.3 can also be worded to say that the spectral sequence (6.4) collapses at the E^2-term. We also note that this identification gives us a choice of representatives for the generators for the cohomology of D_8, but we will see that in most cases the spectral sequence (6.4) has only one nontrivial column or row at E^2, forcing our hand.

Before proving Proposition 5.7, we note that by Lemma 5.3 and Proposition 6.1 we have

$$\alpha_* \Psi^{D_8} = \Psi^{D_8} \alpha_*. \tag{7.4}$$

Also, combining Lemma 5.3 and Proposition 6.6, we have

Proposition 7.5. *Let $\Delta \le D_8$ be the subgroup $(2,3)D_8(2,3) \cap D_8$. Then the map*

$$\alpha_* : H^*(D_8; \pi_* V^{\otimes 4}) \to H^*(D_8; \pi_* V^{\otimes 4})$$

satisfies the identity

$$\alpha_* = t(\Delta, D_8) c_{(2,3)} r(D_8, \Delta).$$

Now, we proceed to prove Proposition 5.7. To do so we exploit naturality using the representability of homotopy (see [12]) and reduce to universal examples. To this end we fix the following throughout:

$$\begin{aligned} \pi_* K(m) &= \mathbb{F}_2\langle a \rangle, & |a| &= m,\\ \pi_* K(n) &= \mathbb{F}_2\langle b \rangle, & |b| &= n,\\ \pi_* K(p) &= \mathbb{F}_2\langle d \rangle, & |d| &= p,\\ \pi_* K(q) &= \mathbb{F}_2\langle e \rangle, & |e| &= q, \end{aligned}$$

where $m, n, p, q \ge 0$.

Proof of Proposition 5.7. *part* (a): First, since $\Delta = \Sigma_2 \times \Sigma_2$, $H^*(\Delta; \mathbb{F}_2) \simeq \mathbb{F}_2[v_1, v_2]$ where $v_1, v_2 \in H^1(\Delta; \mathbb{F}_2)$ is dual to the elements of $H_1(\Delta; \mathbb{F}_2)$ associated to the generators of Δ. We now summarize a result in [9].

Proposition 7.6. *There exist elements $x, y \in H^1(D_8; \mathbb{F}_2)$ and $z \in H^2(D_8; \mathbb{F}_2)$ such that*

1. $H^*(D_8; \mathbb{F}_2) \simeq \mathbb{F}_2[x, y, z]/(xy)$,
2. $$r(D_8, \Delta)x = v_2,$$ $$r(D_8, \Delta)y = 0,$$ $$r(D_8, \Delta)z = v_1(v_1 + v_2),$$
3. $$t(\Delta, D_8)v_1^m = \sum_{0 \le 2l < m} \binom{m-l-1}{l} x^{m-2l} z^l,$$ $$t(\Delta, D_8)v_2^m = 0.$$

Proof. (1) Follows from Proposition 7.3 plus a determination of extensions which is performed in [1].

(2) Is another calculation done in [1].

(3) Is a computation performed in [9]. □

Proposition 7.7. *Under the homomorphism*

$$\psi^{D_8} \colon \pi_* S^{D_8} K(m) \to H^*(D_8; \mathbb{F}_2),$$
$$\psi^{D_8} \sigma_j \sigma_i(a) = x^{m+i-j} z^{m-i}.$$

Proof. As shown in [9], under the identification of Proposition 7.3, x^r is the element w^r in $H^r(\Sigma_2, H^0(\Sigma_2; \pi_{2m}K(m)^{\otimes 2})^{\otimes 2})$ and z^r is the element $w^r \otimes w^r$ in $H^0(\Sigma_2; H^r(\Sigma_2; \pi_{2m}K(m)^{\otimes 2})^{\otimes 2})$. The result now follows from Propositions 7.1 and 7.3. □

Before getting to our main computation, we need

Lemma 7.8. *Let $N \in \mathbb{Z}$ and $a \ge r \ge 0$. Then*

$$\sum_{0 \le l \le r} \binom{r}{l} \binom{N}{s-l} = \binom{N+r}{s}.$$

Proof. This follows from an easy induction on r using the general Pascal's identity. □

Now, combining (7.4), Propositions 7.5, 7.6, and 7.1 we have

$$\psi^{D_8} \alpha_* \sigma_j \sigma_i(a) = \alpha_*(x^{m+i-j} z^{m-i})$$

$$\begin{aligned}
&= t(\Delta, D_8)c_{(2,3)}r(D_8,\Delta)(x^s z^t)\\
&= t(\Delta, D_8)c_{(2,3)}(v_2^s v_1^t(v_1+v_2)^t)\\
&= t(\Delta, D_8)(v_1^s v_2^t(v_1+v_2)^t)\\
&= t(\Delta, D_8)\left(\sum_{0\le k\le t}\binom{t}{k} v_1^{s+t-k}v_2^{t+k}\right). \qquad (7.9)
\end{aligned}$$

Here we have the identity $s=m+i-j$ and $t=m-i$. We have also slipped in $c_{(2,3)}v_1=v_2$.

Now, by (6.7) and Proposition 7.6, (7.9) becomes

$$\begin{aligned}
&\sum_{0\le k\le t}\binom{t}{k} t(\Delta, D_8)v_1^{s+t-k}v_2^{t+k}\\
&= \sum_{0\le k\le t}\binom{t}{k} x^{t+k}\left(\sum_{0\le 2l<s+t-k}\binom{s+t-k-l-1}{l} x^{s+t-k-2l}z^l\right)\\
&= \sum_{0\le 2l<s+t}\left(\sum_{0\le k<s+t-2l}\binom{t}{k}\binom{s+t-k-l-1}{l}\right) x^{s+2t-2l}z^l\\
&= \sum_{0\le 2l}\left(\sum_{0\le k\le t}\binom{t}{k}\binom{s+t-k-l-1}{l}\right) x^{s+2t-2l}z^l, \qquad (7.10)
\end{aligned}$$

where the last equality follows since $k<s+t-2l$. Now, for each k

$$\binom{s+t-k-l-1}{l} = \binom{s+t-k-l-1}{s+t-k-2l-1} = \binom{-l-1}{s+t-2l-k-1}.$$

Applying Lemma 7.8, we obtain

$$\sum_{0\le k\le t}\binom{t}{k}\binom{s+t-k-l-1}{l} = \binom{t-l-1}{s+t-2l-1}.$$

Thus, (7.10) becomes

$$\sum_{0\le 2l<s+t}\binom{t-l-1}{s+t-2l-1} x^{s+t-2l}z^l = \sum_{j<2s}\binom{s-i-1}{2s-j-1} x^{n-i-j+2s}z^{n-s}$$

upon letting $l=n-s$.

Combining Proposition 6.1, (7.4), Propositions 7.5 and 7.7 we arrive at our desired result.

Proof of Proposition 5.7. *parts* (b) *and* (c): As before, it is sufficient to prove it for the case $V=K(m)\times K(n)$.

Let N be the Σ_4-submodule of $\pi_*(K(m)\times K(n))^{\otimes 4}$ generated by $a\otimes a\otimes b\otimes b$. As such N is a direct summand of $\pi_*(K(m)\times K(n))^{\otimes 4}$ as a Σ_4-module. Further, as a D_8-module

$$N = N_1 \otimes N_2,$$

where N_1 is generated by $a\otimes a\otimes b\otimes b$ and N_2 is generated by $a\otimes b\otimes a\otimes b$. Now, writing the extension of (5.1) as

$$B \rightarrowtail D_8 \twoheadrightarrow \Sigma_2,$$

where $B = \langle (1,2),(3,4)\rangle \simeq \Sigma_2\times\Sigma_2$, then N is a direct sum of two trivial B-modules. Thus by the Kunneth theorem

$$H^*(B;N_1)\simeq H^*(B,\mathbb{F}_2)\oplus H^*(B;\mathbb{F}_2)\simeq \mathbb{F}_2[\zeta_1',\zeta_2']\oplus\mathbb{F}_2[\zeta_1'',\zeta_2''].$$

Here Σ_2 acts by exchanging summands, which is a free Σ_2-action. Hence, (6.4) implies that

$$H^*(D_8;N_1)\simeq H^0(\Sigma_2;H^*(B;N_1))\simeq \mathbb{F}_2[\zeta_1,\zeta_2],$$

where ζ_1 corresponds to $\zeta_1'\oplus\zeta_1''$ and ζ_2 corresponds to $\zeta_2'\oplus\zeta_2''$, $|\zeta_1| = 1 = |\zeta_2|$.

Next, N_2 is a free B-module so by (6.4)

$$H^*(D_8;N_2)\simeq H^*(\Sigma_2;H^0(B;N_2))\simeq \mathbb{F}_2[\xi]$$

with $|\xi| = 1$.

Now, we have an extension

$$\Sigma_2 \rightarrowtail \Delta \twoheadrightarrow \Sigma_2$$

so that

$$H^*(\Delta;N_1)\simeq H^0(\Sigma_2;H^*(\Sigma_2;N_1))\simeq \mathbb{F}_2[\eta], \quad |\eta| = 1,$$

since N_1 is a direct sum of two trivial Σ_2-modules with respect to the inner Σ_2-action and so proceed as above. Now, N_2 factors into $N_2'\oplus N_2''$ as Δ-modules where N_2' is generated by $a\otimes b\otimes a\otimes b$ and N_2'' is generated by $a\otimes b\otimes b\otimes a$.

Thus

$$H^*(\Delta;N_2)\simeq \mathbb{F}_2[\lambda_1]\oplus\mathbb{F}_2[\lambda_2] \quad |\lambda_i| = 1 \quad i = 1,2$$

by a computation as above.

Proposition 7.11. (1) *Under the map* $r(D_8,\Delta): H^*(D_8;N)\to H^*(\Delta;N)$

$$\zeta_1 \to \eta,$$
$$\zeta_2 \to \eta,$$
$$\xi \to \lambda_1\oplus\lambda_2.$$

(2) *Under the map* $c_{(2,3)}: H^*(\Delta; N) \to H^*(\Delta; N)$

$$\eta \to \lambda_1,$$
$$\lambda_2 \to \lambda_2.$$

(3) *Under the map* $t(\Delta, D_8): H^*(\Delta; N) \to H^*(D_8; N)$

$$\eta^r \to 0,$$
$$\lambda_1^r \to \zeta^r,$$
$$\lambda_2^r \to \xi^r$$

for all $r > 0$.

Proof. (1) Consider the diagram of extensions

$$\begin{tikzcd} \Sigma_2 \arrow[r, tail] \arrow[d, "\delta"'] & \Delta \arrow[r, two heads] \arrow[d] & \Sigma_2 \arrow[d, equal] \\ B \arrow[r, tail] & D_8 \arrow[r, two heads] & \Sigma_2 \end{tikzcd}$$

where δ is the diagonal map. This induces

$$H^0(\Sigma_2; H^*(B; N_1)) \to H^0(\Sigma_2; H^*(\Sigma_2; N_1))$$

and

$$H^*(\Sigma_2; H^0(B; N_2)) \to H^*(\Sigma_2; H^0(\Sigma_2; N_2)).$$

These are the restriction maps

$$H^*(D_8; N_i) \to H^*(\Delta; N_i)$$

for $i = 1, 2$, by our above computations and Lemma 6.5. The first restriction is an easy computation. For the second restriction we have $H^0(B; N_2) \simeq \mathbb{F}_2$ and $H^0(\Sigma_2; N_2) \simeq \mathbb{F}_2 \oplus \mathbb{F}_2$ so that the induced map $\mathbb{F}_2 \to \mathbb{F}_2 \oplus \mathbb{F}_2$ is the diagonal map.

(2) This is an easy consequence of the fact that

$$c_{(2,3)} N_1 = N_2',$$
$$c_{(2,3)} N_2'' = N_2''.$$

(3) First, N_2 is a free B-module so that $r(D_8, B)$ is trivial on $H^*(D_8; N_2)$ in positive degrees. Next, N_1 is a direct sum of two trivial B-modules thus

$$H^*(B; N_2) \simeq \mathbb{F}_2[\gamma_1, \gamma_2] \oplus \mathbb{F}_2[\varphi_1, \varphi_2]$$

where $|\gamma_i| = 1 = |\varphi_i|$, $i = 1, 2$. From the diagram of extensions

$$\begin{tikzcd} B \arrow[r, tail] \arrow[d, equal] & B \arrow[r, two heads] \arrow[d] & 1 \arrow[d] \\ B \arrow[r, tail] & D_8 \arrow[r, two heads] & \Sigma_2 \end{tikzcd}$$

and Lemma 6.5, the restriction map $r(D_8,B)$ on $H^*(D_8;N_1)$ is equal to the inclusion

$$H^0(\Sigma_2;H^*(B;N_1))\to H^*(B;N_1).$$

Thus

$$r(D_8;B)\zeta_1^s\zeta_2^t=\gamma_1^s\gamma_2^t\oplus\varphi_1^s\varphi_2^t.$$

We now pause to bring in the transfer

Claim.

$$r(D_8,B)t(\Delta,D_8)=0.$$

Proof. By Proposition 6.6

$$r(D_8,B)t(\Delta,D_8)=t(I,B)r(\Delta,I),$$

where

$$I=\Delta\cap B.$$

Since I is a factor of $B, r(B,I)$ is onto, but $t(I,B)r(B,I)=0$ so that $t(I,B)=0$. □

From this claim and our computations, we conclude that

$$t(\Delta,D_8)\lambda_i^r=c_i\xi^r,$$

$c_i\in\mathbb{F}_2$, $i=1,2$. From Proposition 6.6, we have

$$r(D_8,\Delta)t(\Delta,D_8)=1+c_{(1,2)}.$$

Since

$$(1,2)N_1=N_1,$$
$$(1,2)N_1'=N_1'',$$

we get that under $r(D_8,\Delta)t(\Delta,D_8)$

$$\lambda_i^r\to\lambda_1^r\oplus\lambda_2^r.$$

So $c_i=1$ for $i=1,2$. Finally, $t(\Delta,D_8)\eta^r=0$ since η^r is in the image of $r(D_8,\Delta)$. □

Now, the relevance of the module N comes from

Proposition 7.12. (1) *For* $0\le i\le m,\ 0\le j\le n,$

$$\Psi^{D_8}[\sigma_i(a),\sigma_j(b)]=\zeta_1^{m-i}\zeta_2^{n-j}\in H^*(D_8;N_1).$$

(2) *For* $0 \le i \le m+n$

$$\Psi^{D_8}\sigma_i[a,b] = \xi^{n+m-i} \in H^*(D_8; N_2).$$

Proof. These follow from Propositions 7.1–7.3. □

Combining Corollary 6.3, (7.4), Propositions 7.5, 7.11, and 7.12 gives us our desired result.

Proof of Proposition 5.7. *part* (d): Again it is sufficient to prove the result for $V = K(m) \times K(n) \times K(p)$. Let N be the Σ_4-submodule of $\pi_* V^{\otimes 4}$ generated by $a \otimes a \otimes b \otimes d$. As such it is a summand of the Σ_4-module $\pi_* V^{\otimes 4}$.

Proposition 7.13. *For all* $0 \le i \le m$

$$\Psi^{D_8}[\sigma_i(a), [b,d]] \in H^{m-i}(D_8; N).$$

Proof. Again, this is a computation utilizing Propositions 7.1–7.3. □

Now, since N is a free Δ-module, then by (7.4) and Proposition 7.5 the result follows from a computation utilizing Proposition 3.13 and Corollary 5.5.

Proof of Proposition 5.7. *part* (e): Let $V = K(m) \times K(m) \times K(p) \times K(q)$ and N the Σ_4-submodule of $\pi_* V^{\otimes 4}$ generated by $a \otimes b \otimes d \otimes e$.

Proposition 7.14.

$$\Psi^{D_8}[[a,b],[d,e]] \in H^*(D_8; N).$$

Proof. Combine Propositions 7.2 and 7.3. □

N is Σ_4-free so another computation using Proposition 3.13 and Corollary 5.5 gives us our result.

This completes the proof of Proposition 5.7.

Proof of Proposition 5.10. (a) It is sufficient to prove the result for $V = K(n)$ and $W = K(n)$. Suppose $n, m > 0$. Then

$$(N_V \otimes N_W)_* : \pi_* S_2 V \otimes S_2 W \to \pi_* S^2 V \otimes S^2 W$$

is injective. Thus, it suffices to compute $\overline{\alpha}''_*$. By Theorem 6.1 and Lemma 5.9 our conclusion follows from $t(\Sigma_2, \Sigma_2 \times \Sigma_2) = 0$ since $r(\Sigma_2 \times \Sigma_2, \Sigma_2)$ is onto $H^*(\Sigma_2; \pi_* (V \otimes W)^{\otimes 2})$. Suppose $n = 0$. Define

$$i_1 : (S_2 V) \otimes W \to S_2(V \otimes W)$$

as the unique simplicial map such that

$$(xy) \otimes b \to (x \otimes b)(y \otimes b).$$

Also define

$$i_2 : (S_2 V) \otimes W \to (S_2 V) \otimes (S_2 W)$$

as $1 \otimes \iota$ (see (2.6)). Then the diagram

$$\begin{tikzcd} & S_2(V \otimes W) \arrow[dd, "\bar{\phi}''"] \\ (S_2 V) \otimes W \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & \\ & S_2 V \otimes S_2 W \end{tikzcd}$$

commutes. A computation gives the result. The case of $m = 0$ is the same.

(b) This is an easy computation using the diagram

$$\begin{tikzcd} (V \otimes W)^{\otimes 2} \arrow[r] \arrow[d, "1 \otimes T \otimes 1"'] & S_2(V \otimes W) \arrow[d, "\bar{\phi}''"] \\ (V^{\otimes 2}) \otimes (W^{\otimes 2}) \arrow[r] & S_2 V \otimes S_2 W \end{tikzcd} . \qquad \square$$

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